Can you take Komjath's Inaccessible Away?

Hossein Lamei Ramandi

IPM

Tehran, Winter 2025

Introduction

Theorem (Jensen)

There is a Kurepa tree in the constructible universe L*, which has no Aronszajn subtrees.*

Theorem (Todorcevic)

There is a countably closed forcing which adds a Kurepa tree with no Aronszajn subtree.

Theorem (Komjath)

It is consistent relative to the existence of two inaccessible cardinals that there is a Kurepa tree and every Kurepa tree has an Aronszajn subtree.

Introduction

Recall that if it is consistent that there are no Kurepa trees then it is consistent that there is an inaccessible cardinal. In fact, the same assumption implies that ω_2 is inaccessible in the constructible universe.

Question

Assume there are Kurepa trees and every Kurepa tree has an Aronszajn subtree. Is it consistent that there are at least two inaccessible cardinals.

Introduction–Theorems

Theorem (L., Todorcevic)

Assume there is an inaccessible cardinal. Then it is consistent that there is a Kurepa tree and every Kurepa tree contains an Aronszajn subtree.

Theorem (L., Todorcevic)

It is consistent that there is a Kurepa tree T such that whenever $U \subset T$ is a Kurepa tree when it is considered with the inherited order from T, then U has an Aronszajn subtree.

Corollary

Assume MA_{ω_2} holds and ω_2 is not a Mahlo cardinal in L. Then there is a Kurepa tree with the property that every Kurepa subset has an Aronszajn subtree.

Walks on ordinals in ω_2

Definition

A sequence $\langle C_\alpha : \alpha \text{ is limit and } \omega_1 < \alpha < \omega_2 \rangle$ is said to be a $\square_{\omega_1}$-*sequence* if

- C_α is a closed unbounded subset of α,
- $\mathrm{otp}(C_\alpha) < \alpha$ and
- if α is a limit point of C_β then $C_\beta \cap \alpha = C_\alpha$.

The assertion that there is a $\square_{\omega_1}$-sequence is called $\square_{\omega_1}$.

Proposition

If $\square_{\omega_1}$ holds then there is a sequence $\langle C_\alpha : \alpha \in \omega_2 \rangle$ such that

- C_α is a closed unbounded subset of α,
- $C_{\alpha+1} = \{\alpha\}$,
- $\mathrm{otp}(C_\alpha) \leq \omega_1$ and if $\mathrm{cf}(\alpha) = \omega$ then $\mathrm{otp}(C_\alpha) < \omega_1$,
- if $\alpha \in C_\beta$ and β is limit then $\mathrm{cf}(\alpha) \leq \omega$,
- if α is a limit point of C_β then $C_\beta \cap \alpha = C_\alpha$.

Walks

Fact

Assume λ is a regular cardinal which is not Mahlo in L. Let $G \subset \mathrm{coll}(\omega_1, < \lambda)$ be L-generic. Then $\square_{\omega_1}$ holds in $\mathrm{L}[G]$.

Definition

The function $\rho : [\omega_2]^2 \longrightarrow \omega_1$ is defined recursively as follows: for $\alpha < \beta$,

$$\rho(\alpha, \beta) = \max\{\mathrm{otp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\}.$$

We define $\rho(\alpha, \alpha) = 0$ for all $\alpha \in \omega_2$. When the order between α, β is not known we use $\rho\{\alpha, \beta\}$ instead of $\rho(\alpha, \beta)$. More precisely, $\rho\{\alpha, \beta\} = \rho(\alpha, \beta)$ if $\alpha \leq \beta$ and $\rho\{\alpha, \beta\} = \rho(\beta, \alpha)$ if $\beta \leq \alpha$.

Lemma

Assume $\xi \in \alpha$ and α is a limit point of C_β. Then $\rho(\xi, \alpha) = \rho(\xi, \beta)$.

Lemma

If $\alpha < \beta$, α is a limit ordinal such that there is a cofinal sequence of $\xi \in \alpha$, with $\rho(\xi, \beta) \leq \nu$ then $\rho(\alpha, \beta) \leq \nu$.

Lemma

For all $\nu \in \omega_1$ and $\alpha \in \omega_2$, the set $\{\xi \in \alpha : \rho(\xi, \alpha) \leq \nu\}$ is countable.

Lemma

Assume $\alpha \leq \beta \leq \gamma$. Then

- $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,
- $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

Lemma

Assume $\alpha < \beta < \gamma$. We have $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$, if $\rho(\beta, \gamma) < \max\{\rho(\alpha, \beta), \rho(\alpha, \gamma)\}$.

Lemma

Assume $\beta \in \lim(\omega_2)$, and $\gamma > \beta$. Then there is $\beta' \in \beta$ such that for all $\alpha \in (\beta', \beta)$, $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$.

Lemma

Assume A is an uncountable family of finite subsets of ω_2 and $\nu \in \omega_1$. Then there is an uncountable $B \subset A$ such that B forms a Δ-system with root r and for all a, b in B:

- $\rho\{\alpha, \beta\} > \nu$ for all $\alpha \in a \setminus b$ and $\beta \in b \setminus a$,
- $\rho\{\alpha, \beta\} \geq \min\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$ for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$.

Our Canonical Tree

Definition

Assume $A \subset \omega_2$. Q_A is the poset consisting of all finite functions p such that the following holds.

1. $\mathrm{dom}(p) \subset A$.
2. For all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [\omega_1]^{<\omega}$ such that for all $\nu \in \omega_1$, $p(\alpha) \cap [\nu, \nu + \omega)$ has at most one element.
3. For all α, β in $\mathrm{dom}(p)$, $p(\alpha) \cap p(\beta)$ is an initial segment of both $p(\alpha)$ and $p(\beta)$.
4. For all $\alpha < \beta$ in $\mathrm{dom}(p)$, $\max(p(\alpha) \cap p(\beta)) < \rho(\alpha, \beta)$ or $p(\alpha) \cap p(\beta) = \emptyset$.

We let $q \leq p$ if $\mathrm{dom}(p) \subset \mathrm{dom}(q)$ and $\forall \alpha \in \mathrm{dom}(p)$, $p(\alpha) \subset q(\alpha)$. We use Q in order to refer to Q_{ω_2}.

Lemma

Assume $A \subset \omega_2$. The set $\{b_\xi : \xi \in A\}$ is the set of all cofinal branches of the generic tree T in the forcing extension by Q_X.

Complete Suborders of Q

Lemma

The poset Q_c is a complete suborder of Q. Moreover, if $X \subset \omega_2$ is a set of ordinals of cofinality ω_1, then $Q_{\omega_2 \setminus X}$ is a complete suborder of Q.

Lemma

Assume CH. Let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain of countable elementary submodels of H_θ where θ is a regular large enough cardinal, $N_{\omega_1} = \bigcup_{\xi \in \omega_1} N_\xi$, and $\mu = \sup(N_{\omega_1} \cap \omega_2)$. Then Q_μ is a complete suborder of Q.

Lemma

Assume $\mu \in \omega_2$, $x \subset \omega_2$ is finite and $Q_\mu \lhd Q$. Then $Q_\mu \lhd Q_{\mu \cup x} \lhd Q$.

Fact

Assume $\mathrm{cf}(\mu) = \omega$, $\mu \in \omega_2$, for some $\beta > \mu$, μ is a limit point of C_β and the set of all limit points of C_μ is cofinal in μ. Then Q_μ is not a complete suborder of Q.

A way of finding Aronszajn subtrees

Definition

Assume T is an ω_1-tree, κ is a large enough regular cardinal, $t \in T \cup \mathcal{B}(T)$, and $N \prec H_\kappa$ is countable such that $T \in N$. We say that N *captures* t if there is a chain $c \subset T$ in N which contains all elements of $T_{<N\cap\omega_1}$ below t, or equivalently $t \restriction (\delta_N) \subset c$.

Definition

Assume $T = (\omega_1, <)$ is an ω_1-tree, $x \in T \cup \mathcal{B}(T)$ and $N \prec H_\theta$ is countable with $T \in N$. We say that x is *weakly external to* N if there is a stationary $\Sigma \subset [H_{(2^{\omega_1})^+}]^\omega$ in N such that

$$\forall M \in N \cap \Sigma,\ M \text{ does not capture } x.$$

A way of finding Aronszajn subtrees

Proposition

Let $T = (\omega_1, <)$ be an ω_1-tree, $\kappa = (2^{\omega_1})^+$ and $\Sigma \subset [H_\kappa]^\omega$ be stationary. Assume for all large enough regular cardinal θ there are $x \in T$ and countable $N \prec H_\theta$ such that x is weakly external to N, witnessed by Σ. In other words, for all $M \in \Sigma \cap N$, M does not capture x. Then T has an Aronszajn subtree.

Corollary

Assume $T = (\omega_1, <)$ is an ω_1-tree. Then the following are equivalent:

- T has an Aronszajn subtree.
- For all large enough regular cardinal θ there are $x \in T$ and countable $N \prec H_\theta$ such that x is external to N.
- For all large enough regular cardinal θ there are $x \in T$ and countable $N \prec H_\theta$ such that x is weakly external to N.

An inequality for ρ

Lemma

Let $(2^{\omega_1})^+ < \kappa_0 < \kappa < \theta$ be regular cardinals such that $(2^{\kappa_0})^+ < \kappa$, and $(2^\kappa)^+ < \theta$. Let S be the set of all $X \in [\omega_2]^\omega$ such that $C_{\alpha_X} \subset X$ and $\lim(C_{\alpha_X})$ is cofinal in X. Assume $\mathcal{A}$ is the set of all countable $N \prec H_\theta$ with the property that if $N \cap \omega_2 \in S$ then there is a club of countable elementary submodels $E \subset [H_{\kappa_0}]^\omega$ in N such that for all $M \in E \cap N$,

$$\rho(\alpha_M, \alpha_N) \leq M \cap \omega_1.$$

Then $\mathcal{A}$ contains a club.

Theorem

It is consistent that there is a Kurepa tree T such that whenever $U \subset T$ is Kurepa then U has an Aronszajn subtree.

Proof of the Theorem

Assume G is a generic filter for the forcing Q, and T is the tree introduced by G.
Since Q is ccc, it preserves all cardinals and T is a Kurepa tree.
Assume U is a Kurepa subset of T, and X is the set of all $\xi \in \omega_2$ such that $b_\xi \cap U$ is uncountable.
Let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain of countable elementary submodels of H_θ such that $U \in N_0$ and for all $\xi \in \omega_1$, $N_\xi \in \mathcal{A}$, where $\mathcal{A}$ is the same club as in the previous lemma.
Let $N_{\omega_1} = \bigcup_{\xi \in \omega_1} N_\xi$, $\mu = N_{\omega_1} \cap \omega_2$.
Fix $\eta \in X$ above μ.
By Proposition above, it suffices to show that for some $\xi \in \omega_1$, the first element of $b_\eta \cap U$ whose height is more than $N_\xi \cap \omega_1$ is weakly external to N_ξ witnessed by some stationary set Σ.
Without loss of generality we can assume that for all $\xi \in \omega_1$:

- $\alpha_{N_\xi} = \sup(N_\xi \cap \omega_2)$ is a limit point of C_μ,
- $N_\xi \cap \omega_2 \supset C_{\alpha_{N_\xi}}$ and
- $\lim(C_{\alpha_{N_\xi}})$ is a cofinal in α_{N_ξ}.

Proof of the Theorem

Let $\xi \in \omega_1$ be such that $\operatorname{otp}(C_{\alpha_{N_\xi}}) > \rho(\mu, \eta)$ and for all $\zeta > \xi$, $\rho(\alpha_{N_\zeta}, \eta) > \rho(\mu, \eta)$.
Then note that $\rho(\mu, \eta) \in N_\xi$.
Use Lemma to find $E \in N_\xi$ which is a club of countable elementary submodels of H_{ω_3} such that for all $M \in E \cap N_\xi$, $\rho(\mu, \eta) \in M$ and $\rho(\alpha_M, \alpha_{N_\xi}) \leq M \cap \omega_1$.
Now let Σ be the set of all $M \in E$ such that $M \cap \omega_2 \supset C_{\alpha_M}$ and $\lim(C_{\alpha_M})$ is a cofinal subset of α_M.
Routine to show that Σ is stationary and in N_ξ.
Let $M \in \Sigma \cap N_\xi$. We want to show that M does not capture b_η, as a branch of T. Equivalently, for all $b \in M$ which is a cofinal branch of T, $\Delta(b, b_\eta) \in M$.
By the lemma above, it suffices to show that for all $\gamma \in M$, $\rho(\gamma, \eta) \leq M \cap \omega_1$.
Recall that:

$$\rho(\gamma, \eta) \leq \max\{\rho(\gamma, \alpha_M), \rho(\alpha_M, \mu), \rho(\mu, \eta)\}.$$

Lemma

Assume $\omega_1 < \mu < \omega_2$, $Q_\mu \lhd Q$ in V and $G \subset Q$ is V-generic. Let K be an ω_1-tree in $\mathrm{V}[G_\mu]$ and $b \subset K$ be a cofinal branch in $\mathrm{V}[G]$. Then there is a finite $x \subset [\mu, \omega_2)$ such that $b \in \mathrm{V}[G_{\mu \cup x}]$.

Lemma (Consequence of Baumgartner, Malitz, Reinhardt.)

For every Aronszajn tree A there is a ccc poset which adds an antichain to A and which does not add new cofinal branches to the ω_1-trees of the ground model.

Notation

– If $G \subset Q$ is a generic filter and $X \subset \omega_2$ with $Q_X \lhd Q$, $G_X = G \cap Q_X$.
– If $Q_X \lhd Q_A \lhd Q$, $R_{X,A}$ refers to the ccc poset such that $Q_A = Q_X * \dot{R}_{X,A}$.
– $R_{X,A}$ can be described more explicitly in the forcing extension by G_X as follows. Let T be the generic tree for Q_X and b_ξ be the set of all $t \in T$ such that $t \in q(\xi)$ for some $q \in G_X$. Recall that b_ξ is an uncountable downward closed branch of T. Moreover, every branch of T in the forcing extension by G_X has to be b_ξ for some $\xi \in X$. The poset $R_{X,A}$ consists of finite partial functions p from $A \setminus X$ to T such that:

1. for every $\alpha \in \mathrm{dom}(p)$ and $\xi \in X$, $(p(\alpha) \wedge b_\xi) < \rho\{\xi, \alpha\}$ and
2. for all $\alpha < \beta$ in $\mathrm{dom}(p)$, $(p(\alpha) \wedge p(\beta)) < \rho(\alpha, \beta)$.

In $R_{X,A}$, $q \leq p$ if $\mathrm{dom}(q) \supset \mathrm{dom}(p)$ and $p(\alpha) \leq_T q(\alpha)$ for all $\alpha \in \mathrm{dom}(p)$. We sometimes use the notation $R_A(B)$ in order to refer to $R_{A,A\cup B}$ if A, B are disjoint.

Finiteness Lemma

Lemma

Assume $\omega_1 < \mu < \omega_2$, $Q_\mu \lhd Q$ *in* V *and* $G \subset Q$ *is* V*-generic. Let* K *be an* ω_1*-tree in* $\mathrm{V}[G_\mu]$ *and* $b \subset K$ *be a cofinal branch in* $\mathrm{V}[G]$*. Then there is a finite* $x \subset [\mu, \omega_2)$ *such that* $b \in \mathrm{V}[G_{\mu \cup x}]$.

Proof:

– Work in $\mathrm{V}[G_\mu]$.

– Let $r \in R_{\mu,\omega_2} \cap G$, and $\tau \subset K \times \{q \in R_{\mu,\omega_2} : q \leq r\}$ be an R_{μ,ω_2}-name.

– Assume for a contradiction that for any finite $x \subset \omega_2$,

$$r \Vdash_{R_{\mu,\omega_2}} \text{“}\tau \text{ is a cofinal branch of } K \text{ outside } \mathrm{V}[G_\mu * \dot{H}_x]\text{.”} \tag{1}$$

Proof of finiteness Lemma

– Without loss of generality $\tau = \bigcup\{\{u\} \times E_u : u \in K\}$ such that:

1. E_u is an antichain that is maximal in $C_u = \{q \leq r : q \Vdash_{R_{\mu,\omega_2}} u \in \tau\}$.
2. If $q \in E_u$ and $\alpha \in \mathrm{dom}(q)$ then $\mathrm{ht}_T(q(\alpha)) \geq \mathrm{ht}_K(u)$.
3. If $q \in E_u$ then q is a one-to-one function whose range consists of the elements of the same height in T.

– R_{μ,ω_2} is ccc so E_u is countable.
– $U = \mathrm{dom}(\tau)$ is a Souslin subtree of K.
– Fix uncountable $\Gamma \subset \mathrm{range}(\tau)$ such that $\{\mathrm{dom}(p) : p \in \Gamma\}$ forms a Δ-system with root w and for some $k \in \omega$ elements of Γ have size $k + |w|$.
– Note that $r \in R_\mu(w)$.

Proof of Finiteness Lemma

We showed it is enough to find $H_w \subset R_\mu(w), P$ such that:

1. $H_w \subset R_\mu(w)$ is $\mathrm{V}[G_\mu]$-generic and $r \in H_w$.
2. P is a ccc poset in $\mathrm{V}[G_\mu * H_w]$ and it does not add new cofinal branches to the ω_1-trees of the ground model.
3. In $\mathrm{V}[G_\mu * H_w]^P$ there is an uncountable $\mathcal{A}' \subset \Gamma$ whose elements are p.w. compatible.
4. every element in $\mathcal{A}'$ is compatible with every element in $G_\mu * H_w$.

– Define $\mathcal{A} = \{p \upharpoonright (\mathrm{dom}(p) \setminus w) : p \in \Gamma \wedge p \upharpoonright w \in G_{\mu \cup w}\}$.
– We also showed that we are done if $\mathcal{A}$ is countable. From now on assume $\mathcal{A}$ is uncountable.
– Note that every element of $\mathcal{A}$ is compatible with every element in $G_{\mu \cup w}$.

Proof of Finiteness Lemma

– For each $p \in \mathcal{A}$, let $d_p : k \longrightarrow \mathrm{dom}(p)$ be the unique strictly increasing bijection.

– Let $\langle I_l : 0 < l \leq \frac{k(k+1)}{2} + 1 = m \rangle$ be a sequence listing all $I \subset k$ with $0 < |I| \leq 2$ such that all singletons are listed before pairs.

– We are going to find $\langle \mathrm{V}_l, \mathcal{A}_l, P_l : l \leq \frac{k(k+1)}{2} + 1 \rangle$, by induction on l, such that:

- $\mathrm{V}_0 = \mathrm{V}[G_{\mu \cup w}]$.
- $P_l \in \mathrm{V}_l$ is ccc and does not add cofinal branches to ω_1-trees of V_l.
- $\mathrm{V}_{l+1} = \mathrm{V}_l[P_l]$.
- $\mathcal{A}_l \in \mathrm{V}_l$ is uncountable for all l.
- $\mathcal{A}_{l+1} \subset \mathcal{A}_l \subset \mathcal{A}_0 = \mathcal{A}$.
- If $\{p, q\} \subset \mathcal{A}_l$ then $p \restriction \{d_p(n) : n \in I_l\}$ and $q \restriction \{d_q(n) : n \in I_l\}$ are compatible in $R_{\mu \cup w, \omega_2}$.

– Then $\mathcal{A}' = \{p \in \Gamma : p \restriction (\mathrm{dom}(p) \setminus w) \in \mathcal{A}_m \wedge p \restriction w \in G_w\}$ works.

Proof of Finiteness Lemma

We proceed by finding $\mathrm{V}_l, \mathcal{A}_l, P_{l-1}$ when $\mathrm{V}_{l-1}, \mathcal{A}_{l-1}, I_l$ are given. First assume $0 < l \leq k$, which means $I_l = \{n\}$ for some $n \in k$. This task can be done by managing the following cases:

1. The map $p \mapsto p(d_p(n))$ is constant on some uncountable subset of $\mathcal{A}_{l-1}$.
2. The map $p \mapsto p(d_p(n))$ is countable-to-one and the downward closure of $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ has an uncountable branch.
3. The map $p \mapsto p(d_p(n))$ is countable-to-one and the downward closure of $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ is Aronszajn.

Case 1, $|I_l| = 1$

fix uncountable $\mathcal{B} \subset \mathcal{A}_{l-1}$ such that $p \mapsto p(d_p(n))$ is constant on $\mathcal{B}$. Let $\nu = p(d_p(n))$ for some (any) $p \in \mathcal{B}$. Let $\mathcal{A}_l \subset \mathcal{B}$ be uncountable such that if $p \neq q$ are in $\mathcal{A}_l$ then $\rho\{d_p(n), d_q(n)\} > \nu$. It is easy to see that $\mathcal{A}_l$ together with $\mathrm{V}_l = \mathrm{V}_{l-1}$ works.

Case 2, $|I_l| = 1$

Let W be the downward closure of $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T. By the lemmas above let $\xi \in \mu \cup w$ such that $b_\xi \subset W$.
Let $\langle p_i : i \in \omega_1 \rangle$ be a sequence in $\mathcal{A}_{l-1}$ such that $\langle p_i(d_{p_i}(n)) \wedge b_\xi : i \in \omega_1 \rangle$ is strictly increasing.
Let $\Gamma_0 \subset \omega_1$ be uncountable such that $\langle \alpha_i = d_{p_i}(n) : i \in \Gamma_0 \rangle$ and $\langle \rho\{\alpha_i, \xi\} : i \in \Gamma_0 \rangle$ are both strictly increasing.
Recall that $\rho\{\alpha_i, \xi\} \geq b_\xi \wedge p_i(\alpha_i)$, so this is possible. Find uncountable $\Gamma_1 \subset \Gamma_0$ such that $\rho(\alpha_i, \alpha_j) \geq \min\{\rho\{\alpha_i, \xi\}, \rho\{\alpha_j, \xi\}\}$ for $i < j$ in Γ_1. In order to see $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$ and $\mathrm{V}_l = \mathrm{V}_{l-1}$ work, assume for a contradiction that $p_i(\alpha_i) \wedge p_j(\alpha_j) \geq \rho(\alpha_i, \alpha_j)$ for some $i < j$ in Γ_1. Then

$$\rho\{\xi, \alpha_i\} > p_i(\alpha_i) \wedge b_\xi = p_i(\alpha_i) \wedge p_j(\alpha_j) \geq \rho(\alpha_i, \alpha_j) \geq \rho\{\alpha_i, \xi\},$$

which obviously is a contradiction.

Case 3, $|I_l| = 1$

let W be a pruned downward closed uncountable subtree of the downward closure of $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T.
Let P_{l-1} be the ccc poset consisting of finite antichains of W, and $\mathrm{V}_l = \mathrm{V}_{l-1}^{P_{l-1}}$. From now on we work in V_l. Fix $\gamma > \sup\{d_p(n) : p \in \mathcal{A}_{l-1}\}$ in ω_2 and $\langle t_i : i \in \omega_1 \rangle$ in A such that if $i < j$ then $\mathrm{ht}(t_i) < \mathrm{ht}(t_j)$. Since W is pruned, for every $t \in W$ there are uncountably many p in $\mathcal{A}_{l-1}$ with $t \leq_T p(d_p(n))$. Since ω_2 is preserved, the square sequence of V_{l-1} is a square sequence in V_l. Therefore, for each $i \in \omega_1$ there is $p_i \in \mathcal{A}_{l-1}$ such that $t_i \in \rho(d_{p_i}(n), \gamma)$ and $t_i <_T p_i(d_{p_i}(n))$. Let $\alpha_i = d_{p_i}(n)$. Find uncountable $\Gamma_0 \subset \omega_1$ such that $\langle \alpha_i : i \in \Gamma_0 \rangle$ and $\langle \rho(\alpha_i, \gamma) : i \in \Gamma_0 \rangle$ are both strictly increasing. Find uncountable $\Gamma_1 \subset \Gamma_0$ such that

$$\rho(\alpha_i, \alpha_j) \geq \min\{\rho(\alpha_i, \gamma), \rho(\alpha_j, \gamma)\}$$

whenever $i < j$ in Γ_1. In order to see $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$ works, assume $i < j$ are in Γ_1. Then

$$p_i(\alpha_i) \wedge p_j(\alpha_j) < t_i < \rho(\alpha_i, \gamma) = \min\{\rho(\alpha_i, \gamma), \rho(\alpha_j, \gamma)\} \leq \rho(\alpha_i, \alpha_j),$$

as desired.

$I_l = \{n, m\}$.

Observation. Assume $\mathbb{V}$ is a forcing extension of $\mathrm{V}[G_{\mu \cup w}]$ by a forcing described in Lemma above. Let $m < n < k$ and assume in $\mathbb{V}$, $\mathcal{B} \subset \mathcal{A}$ is uncountable such that the maps $p \mapsto p(d_p(n))$ and $p \mapsto p(d_p(m))$ are countable-to-one on $\mathcal{B}$. Then either

(a) there are incomparable s, t in T and uncountable $\mathcal{B}_0 \subset \mathcal{B}$ such that for all $p \in \mathcal{B}_0$, $s <_T p(d_p(m))$ and $t <_T p(d_p(n))$, or

(b) $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{B}\}$ is uncountable.

$I_l = \{n, m\}$.

(0) At least one of the maps $p \mapsto p(d_p(n))$ or $p \mapsto p(d_p(m))$ is not countable-to-one on $\mathcal{A}_{l-1}$.

(a) There are incomparable s, t in T and uncountable $\mathcal{B}_0 \subset \mathcal{A}_{l-1}$ such that for all $p \in \mathcal{B}_0$, $s <_T p(d_p(m))$ and $t <_T p(d_p(n))$. Moreover, the maps $p \mapsto p(d_p(n))$ and $p \mapsto p(d_p(m))$ are countable-to-one on $\mathcal{A}_{l-1}$.

(b.1) The downward closure of $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T has an uncountable branch and the maps $p \mapsto p(d_p(n))$ and $p \mapsto p(d_p(m))$ are countable-to-one on $\mathcal{A}_{l-1}$.

(b.2) The downward closure of $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T is an Aronszajn tree and the maps $p \mapsto p(d_p(n))$ and $p \mapsto p(d_p(m))$ are countable-to-one on $\mathcal{A}_{l-1}$.

Case 0

The forcing extension is the trivial forcing extension. Find uncountable $\mathcal{B} \subset \mathcal{A}_{l-1}$ and $t \in T$ such that one of the maps $p \mapsto p(d_p(n))$ or $p \mapsto p(d_p(m))$ is constantly t on $\mathcal{B}$. Let $\nu = t + 1$ and let $\mathcal{A}_l \subset \mathcal{B}$ be uncountable such that for $p \neq q$ in $\mathcal{A}_l$, $\rho\{d_p(n), d_q(m)\} > \nu$. So for all distinct p, q in $\mathcal{A}_l$, $p(d_p(n)) \wedge q(d_q(m)) < \nu < \rho\{d_p(n), d_q(m)\}$. By the symmetry and since we have already dealt with the one element subsets of k, this finishes case (0).

Case a

The forcing extension is the trivial forcing extension. Fix $s, t, \mathcal{B}_0$ as in (a) of Observation. Let $\mathcal{A}_l \subset \mathcal{B}_0$ be uncountable such that for $p \neq q$ in $\mathcal{A}_l$, $t < \rho\{d_p(n), d_q(m)\}$. Then for all $p \neq q$ in $\mathcal{A}_l$, $p(d_p(n)) \wedge p(d_q(m)) = s \wedge t < t < \rho\{d_p(n), d_q(m)\}$. Because of symmetry and the fact that we dealt with the one element sets in the previous steps, this finishes case (a).

Case b.1

The forcing extension is the trivial forcing extension. Assume W is the downward closure of the uncountable set $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T. Let $\xi \in \mu \cup w$ such that $b_\xi \subset W$. We can find $\{p_i : i \in \omega_1\} \subset \mathcal{A}_{l-1}$ such that $\langle p_i(d_{p_i}(m)) \wedge p_i(d_{p_i}(n)) \wedge b_\xi : i \in \omega_1\rangle$ is strictly increasing. $\exists \Gamma_0 \in [\omega_1]^{\aleph_1}$ s.t.

- $\langle \alpha_i = d_{p_i}(n) : i \in \Gamma_0\rangle$, $\langle \beta_i = d_{p_i}(m) : i \in \Gamma_0\rangle$,
- $\langle \{(p_i(\alpha_i) \wedge b_\xi), (p_i(\beta_i) \wedge b_\xi)\} : i \in \Gamma_0\rangle$,
- $\langle \{\rho\{\alpha_i, \xi\}, \rho\{\beta_i, \xi\}\} : i \in \Gamma_0\rangle$ are all strictly increasing.

$\exists \Gamma_1 \in [\Gamma_0]^{\aleph_1}$ $\forall i \neq j$ in Γ_1, $\rho\{\alpha_i, \beta_j\} \geq \min\{\rho\{\alpha_i, \xi\}, \rho\{\beta_j, \xi\}\}$,
Assume $i < j$ are in Γ_1. Then

$$p_i(\alpha_i) \wedge p_j(\beta_j) = p_i(\alpha_i) \wedge b_\xi < \rho\{\alpha_i, \xi\} = \min\{\rho\{\alpha_i, \xi\}, \rho\{\beta_j, \xi\}\}.$$

So $p_i(\alpha_i) \wedge p_j(\beta_j) < \rho\{\alpha_i, \beta_j\}$. Again, by the fact that we have already dealt with the one element sets before, $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$ and $\mathrm{V}_l = \mathrm{V}_{l-1}$ works. This finishes case (b.1).

Cae b.2

let W be the downward closure of the uncountable set $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$ in T. Let W' be an uncountable downward closed pruned subtree of W. Let P_{l-1} be the poset consisting of finite antichains of W'. Let V_l be a forcing extension of V_{l-1} by P_{l-1}. Work in V_l and let $\{t_i : i \in \omega_1\} \subset A$ such that $\langle \mathrm{ht}_T(t_i) : i \in \omega_1 \rangle$ is strictly increasing. Let $\gamma \in \omega_2$ be above all ordinals in $\{p(d_p(n)) + p(d_p(m)) : p \in \mathcal{A}_{l-1}\}$. For each $i \in \omega_1$, find $p_i \in \mathcal{A}_{l-1}$ such that

- $t_i <_T (p_i(\alpha_i) \wedge p_i(\beta_i))$ where $\alpha_i = d_{p_i}(n)$ and $\beta_i = d_{p_i}(m)$,
- $t_i \in \rho(\alpha_i, \gamma)$, and $t_i \in \rho(\beta_i, \gamma)$.

This is possible because the maps $p \mapsto p(d_p(n))$ and $p \mapsto p(d_p(m))$ are countable-to-one and W' is pruned. Let $\Gamma_0 \subset \omega_1$ be uncountable such that:

- $\rho\{\alpha_i, \beta_j\} \geq \min\{\rho(\alpha_i, \gamma), \rho(\beta_j, \gamma)\}$ for all distinct i, j in Γ_0, and
- $\langle \{\rho(\alpha_i, \gamma), \rho(\beta_i, \gamma)\} : i \in \Gamma_0 \rangle$ is strictly increasing.

Now we show that $\mathcal{A}_l = \{p_i : i \in \Gamma_0\}$ works. Assume $i < j$ in Γ_0. Then

$$p_i(\alpha_i) \wedge p_j(\beta_j) < t_i \in \rho(\alpha_i, \gamma) = \min\{\rho(\alpha_i, \gamma), \rho(\beta_j, \gamma)\} \leq \rho(\alpha_i, \beta_j).$$

As in the previous case, by symmetry and the fact that we have already dealt with the one element I_l's, this finishes the work for case (b.2).

Proof of Main Theorem

Now we are ready to prove the main theorem. Assume λ is the first inaccessible cardinal in L and V is the generic extension of L by the Levy collapse forcing with countable conditions which makes λ the second uncountable cardinal. Note that V is a model of $\square_{\omega_1}$. Assume $G \subset Q$ is V-generic and $T, \langle b_\xi : \xi \in \lambda \rangle$ are the generic tree and branches that are defined from G as usual. We show for every Kurepa tree K in $\mathrm{V}[G]$ there is a Kurepa subtree of T which club embeds into K. Recall the following two theorems.

Jensen-Schlechta

Lemma

Assume $A \in \mathrm{V}$ is a countably closed poset, $F \subset A$ is V-generic, $B \in \mathrm{V}$ is a ccc poset and $G \subset B$ is $\mathrm{V}[F]$-generic. Let $T \in \mathrm{V}[G]$ be a normal ω_1-tree.

1. *If $b \in \mathrm{V}[F][G]$ is a cofinal branch in T, then $b \in \mathrm{V}[G]$.*
2. *If $S \in \mathrm{V}[F][G]$ is a downward closed Souslin subtree of T then $S \in \mathrm{V}[G]$.*

Lemma

Let $\lambda \in \mathrm{V}$ be an inaccessible cardinal, $F \subset coll(\omega_1, < \lambda)$ be V-generic, $\mathbb{P}$ be a ccc poset of size $\aleph_1$ in $\mathrm{V}[F]$, $G \subset \mathbb{P}$ be $\mathrm{V}[F]$-generic and $U \in \mathrm{V}[F][G]$ be an ω_1-tree. Then U has at most $\aleph_1$ many Souslin subtrees and cofinal branches in $\mathrm{V}^{\mathbb{P}}$.

Abraham-Shelah

Lemma

Assume R and all its derived trees are Souslin, A is an Aronszajn tree and R' is a derived tree of R whose dimension is n. Moreover assume forcing with R' adds a new branch to A and R' has the least dimension with respect to this property among the derived trees of R. Then R' club embeds into A.

Notation

For finite $x \subset [\mu, \omega_2)$, let $S^\mu[x]$ be the set of all $\langle v_i : i \in |x| \rangle \in T^{[|x|]}$ such that for some $q \in R_{\mu,\omega_2}$:

- $\mathrm{dom}(q) \supset x$ and
- for all $i \in |x|$, $q(x(i)) = v_i$.

So in particular every condition in R_{μ,ω_2} force that $\bigotimes_{\alpha \in x} \dot{b}_\alpha \subset S^\mu[x]$.

For $\alpha \in \omega_2 \setminus \mu$, we use $S^\mu[\alpha]$ instead of $S^\mu[\{\alpha\}]$.

Club Embedding

Assume for a contradiction that $K \in \mathrm{V}[G]$ is a Kurepa tree,
$\dot{K}$ is a Q-name for K,
and $p_0 \in G$ forces that
$\dot{K}$ is a Kurepa tree such that no Kurepa subtree of $\dot{T}$ club embeds into $\dot{K}$.
Let $\mu_0 \in \omega_2$ such that $Q_{\mu_0} \lhd Q$, K and T are in $V[G_{\mu_0}]$ and $p_0 \in G_{\mu_0}$. Note that in $\mathrm{V}[G_{\mu_0}]$,

$$R_{\mu_0,\omega_2} \Vdash \text{“no Kurepa subtree of } \check{T} \text{ club embeds into } \check{K}.\text{”} \tag{2}$$

Club Embedding

Let $Y \in \mathrm{V}[G_{\mu_0}]$ be the set of all (τ, p, x, A) such that:

(a_0) x is a finite subset of $[\mu_0, \omega_2)$,

(a_1) τ is an $R_{\mu_0}(x)$-name,

(a_2) $p \Vdash_{R_{\mu_0}(x)}$ "τ is a cofinal branch of $\check{K}$ which is not in $\mathrm{V}[G_{\mu_0} * \dot{H}_{x'}]$, for any finite x' which is a proper subset of x", where $\dot{H}_{x'}$ is the canonical name for the $\mathrm{V}[G_{\mu_0}]$-generic filter of $R_{\mu_0}(x')$,

(a_3) p is a one-to-one function, $\mathrm{dom}(p) = x$ and $\mathrm{range}(p)$ consists of the elements of the same height in T,

(a_4) $A = \{u \in K : \exists q \in R_{\mu_0}(x)\ q \leq p \wedge q \Vdash \check{u} \in \tau\}$.

Club Embedding

– For $i \in \{1, 2, 3, 4\}$ let Y_i be the projection of Y on the i'th component.
– By Jensen-Schlechta and finiteness, $|Y_3| = \aleph_2$.
– Let $\langle x_\xi : \xi \in \omega_2 \rangle$ be an enumeration of Y_3.
– Let $n \in \omega$ and $\Gamma_0 \subset \omega_2$ be of size $\aleph_2$ such that $\{x_\xi : \xi \in \Gamma_0\}$ is a Δ-system with root w and $|x_\xi| = n + |w|$ for $\xi \in \Gamma_0$.
– W.L.G., assume that $\langle y_\xi = x_\xi \setminus w : \xi \in \Gamma_0 \rangle$ is strictly increasing.
– For every $\xi \in \Gamma_0$ let $\tau'_\xi, p'_\xi, A'_\xi$ be such that $(\tau'_\xi, p'_\xi, x_\xi, A'_\xi) \in Y$.
– W.L.G., for all $i \in n + |w|$, $\xi \mapsto p'_\xi(x_\xi(i))$ is constant on Γ_0.
There is a condition $r \in R_{\mu_0,\omega_2}$ which forces that for $\aleph_2$ many $\xi \in \Gamma_0$, p'_ξ is in the generic filter $\dot{H}_{[\mu_0,\omega_2)}$. In order to contradict (2), we need to work with a $\mathrm{V}[G_{\mu_0}]$-generic filter of R_{μ_0,ω_2} which intersects $\{p'_\xi : \xi \in \Gamma_0\}$ on a set of size $\aleph_2$. Due to similarity of arguments and for easier notation let's assume without loss of generality that

$$|G \cap \{p'_\xi : \xi \in \Gamma_0\}| = \aleph_2. \tag{3}$$

Club Embedding

–Fix $\mu \in \omega_2 \setminus \mu_0$ above $\max(w)$ such that $Q_\mu \lhd Q$.
– From now on, we work in $\mathrm{V}[G_\mu]$ unless otherwise stated.
– Define $\Gamma_1 \in \mathrm{V}[G_\mu]$ to be the set of all $\xi \in \Gamma_0$ such that $\min(y_\xi) > \mu$ and $p'_\xi \restriction w \in G_\mu$.
Obviously $|\Gamma_1| = \aleph_2$ by (3).
– For each $\xi \in \Gamma_1$ let $p_\xi = p'_\xi \restriction y_\xi$.
– Note that by (a_3) and the definition of Γ_1, p_ξ is compatible with every condition in G_μ.
Via the natural transition of objects τ'_ξ, A'_ξ from $\mathrm{V}[G_{\mu_0}]$ to $\mathrm{V}[G_\mu]$, we can find τ_ξ, A_ξ in $\mathrm{V}[G_\mu]$ such that for all $\xi \in \Gamma_1$ the statement (a_i) above implies (b_i) below:

(b_1) τ_ξ is an $R_\mu(y_\xi)$-name,

(b_2) $p_\xi \in R_\mu(y_\xi)$ forces that τ_ξ is a cofinal branch of $\check{K}$ which is not in $\mathrm{V}[G_\mu]$,

(b_3) p_ξ is a one-to-one function and the elements in $\mathrm{range}(p_\xi)$ have the same height in T,

(b_4) $A_\xi = \{u \in K : \exists q \in R_\mu(y_\xi)\ q \leq p_\xi \wedge q \Vdash \check{u} \in \tau_\xi\}$.

Club embedding

We only show how we obtain (b_2). Assume for a contradiction that $\xi \in \Gamma_1$, $r \in G_\mu \cap R_{\mu_0,\mu}$ is an extension of $p'_\xi \restriction w$ and $\bar{p}_\xi \in R_{\mu,\omega_2}$ is an extension of p_ξ such that:

$$r * \bar{p}_\xi \Vdash_{R_{\mu_0,\omega_2}} \tau'_\xi \text{ is a cofinal branch in } \mathrm{V}[G_{\mu_0} * \dot{H}_{\mu_0,\mu}].$$

Since $r * \bar{p}_\xi$ extends p'_ξ, by (a_2),

$$r * \bar{p}_\xi \Vdash_{R_{\mu_0,\omega_2}} \tau'_\xi \text{ is a cofinal branch in } \mathrm{V}[G_{\mu_0} * \dot{H}_{\mu_0,\mu}] \cap \mathrm{V}[G_{\mu_0} * \dot{H}_{x_\xi}].$$

This contradicts (a_2) because by Lemma [no-extra-branch], for every $\mathrm{V}[G_{\mu_0}]$-generic filter $H \subset R_{\mu,\omega_2}$, $\mathrm{V}[G_{\mu_0} * H_x] \cap \mathrm{V}[G_{\mu_0} * H_{\mu_0,\mu}] = \mathrm{V}[G_{\mu_0} * H_{x\cap\mu}]$ and $x_\xi \cap \mu$ is a proper subset of x_ξ. Hence (b_2) holds.

Club Embedding

Note that by [Jensen-Schlechta], all finite powers of T and K have at most $\aleph_1$ many cofinal branches and Souslin subtrees in $\mathrm{V}[G_\mu]$. Let $\Gamma_2 \subset \Gamma_1$ be of size $\aleph_2$ such that for all ξ and η in Γ_2 the following hold:

- $S^\mu[y_\xi(i)] = S^\mu[y_\eta(i)]$ for all $i \in n$,
- $S^\mu[y_\xi] = S^\mu[y_\eta]$,
- $A_\xi = A_\eta$.

Observe that if $y \in \{y_\xi : \xi \in \Gamma_2\}$ and $\bar{v} = \langle v_i : i \in n\rangle$ is an element of $S^\mu[y]$, and v_i's are pairwise distinct then $\bigotimes_{i \in n}(S^\mu[y(i)])_{v_i} = (S^\mu[y])_{\bar{v}}$. Moreover, this tree does not depend on the choice of y.

Club Embedding

–For $i \in n$, let $t_i = p_\xi(y_\xi(i))$ for some (any) $\xi \in \Gamma_2$.
– Let $\Gamma_3 \subset \Gamma_2$ with $|\Gamma_3| = \aleph_2$ such that if $\xi < \eta$ are in Γ_3, $\alpha \in y_\xi$, $\beta \in y_\eta$, then $\rho(\alpha, \beta) > \max\{t_i : i \in n\}$.
For every $\zeta \in \Gamma_3$ define φ_ζ from $\bigotimes_{i \in n}(S^\mu[y(i)])_{t_i}$ to the poset consisting of all extensions of $p_\zeta = \{(y_\zeta(i), t_i) : i \in n\}$ in $R_\mu(y_\zeta)$ as follows. For every $\bar{s} = \langle s_i : i \in n \rangle$ in $\bigotimes_{i \in n}(S^\mu[y(i)])_{t_i}$, let $\varphi_\zeta(\bar{s})$ be the function defined on y_ζ which sends $y_\zeta(i)$ to s_i. It is easy to see that φ_ζ is an isomorphism from its domain to a dense subset of the set of all extensions of p_ζ in $R_\mu(y_\zeta)$. Let $S = \bigcup_{i \in n}(S^\mu[y(i)])_{t_i}$ and $U = A_\zeta$. Obviously, U is Souslin in $\mathrm{V}[G_\mu]$.
– $\mathrm{V}[G_\mu]$ thinks that there is a derived tree of S, namely $\bigotimes_{i \in n}(S^\mu[y(i)])_{t_i}$, which adds a branch to U.

Club Embedding

Claim. All derived trees of S are Souslin in $\mathrm{V}[G_\mu]$.

Proof.

Assume $\langle s^i_j : i \in n \wedge j \in m \rangle$ are pairwise distinct elements of S with the same height δ such that $t_i \leq s^i_j$ for all i, j. We will show that $\prod\{S_{s^i_j} : i \in n \wedge j \in m\}$ is the set of all possible points of a branch of $T^{[mn]}$ which is added by a ccc poset in $\mathrm{V}[G_\mu]$. Let $\langle \xi_j : j \in m \rangle$ be a strictly increasing sequence in Γ_3 such that for all $j < k < m$ if $\alpha \in y_{\xi_j}$ and $\beta \in y_{\xi_k}$ then $\rho(\alpha, \beta) > \delta + \omega$. Let $z_j = y_{\xi_j}$. Define $p : \bigcup_{j \in m} z_j \longrightarrow T$ by $p(z_j(i)) = s^i_j$. By the requirement on Γ_3 and the fact that φ_{ξ_j} is an isomorphism, $p \restriction z_j \in R_\mu(z_j)$ for all $j \in m$. The way we chose the z_j's implies that $p \in R_\mu(\bigcup_{j \in m} z_j)$.

Obviously, the set of all extensions of p in $R_\mu(\bigcup_{j \in m} z_j)$ is a ccc poset in $\mathrm{V}[G_\mu]$ and it adds a new branch to $T^{[mn]}$. Observe that the set $\prod\{S_{s^i_j} : i \in n \wedge j \in m\}$ is the set of all possible points of this branch. □

S becomes Kurepa in some generic extensions.

Claim. Assume $\langle v_j : j \in k \rangle$ is a sequence of pairwise distinct elements of the same height in S. Then in $\mathrm{V}[G_\mu]$, there is a condition q in R_{μ,ω_2} which forces that each S_{v_j} is Kurepa.

Proof.

Fix $\Gamma_4 \subset \Gamma_3$ such that $|\Gamma_4| = \aleph_2$ and for all $\xi < \eta$ in Γ_4, for all $\alpha \in y_\xi$, for all $\beta \in y_\eta$,

$$\rho(\alpha, \beta) > \max\{v_i : i \in k\}.$$

For every increasing $\sigma = \langle \xi_l : l \in k \rangle$ in Γ_4, let $q_\sigma : \bigcup_{l \in k} y_{\xi_l} \longrightarrow S$ be a function such that $q_\sigma(y_{\xi_l}(i)) = v_j$ if v_j is the l'th ordinal in $\langle v_j : j \in k \rangle$ that is above t_i in T. If there is no l'th ordinal in $\langle v_j : j \in k \rangle$ that is above t_i in T, let $q_\sigma(y_{\xi_l}(i)) = t_i$.
The same argument as in the previous claim shows that $q_\sigma \in R_{\mu,\omega_2}$.
Let $\Gamma_5 \subset [\Gamma_4]^k$ be a collection of pairwise disjoint sets with $|\Gamma_5| = \aleph_2$. Since R_{μ,ω_2} is ccc, there is a condition $q \in R_{\mu,\omega_2}$ which forces that for $\aleph_2$ many $\sigma \in \Gamma_5$, q_σ is in the generic filter. But then q forces that S_{v_j} is Kurepa for all $j \in k$. □